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Constraint qualification in a general class of Lipschitzian mathematical programming problems

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Abstract The Kuhn–Tucker type necessary optimality conditions are given for the problem of minimizing the sum of a differentiable function and a locally Lipschitzian function subject to a set of differentiable nonlinear inequalities on a convex subset C of $\mathbb{R}^n$, under the condition of a generalized Kuhn–Tucker constraint qualification or a generalized Arrow–Hurwicz–Uzawa constraint qualification. The case when the set C is open is shown to be a special one of our results, which helps us to improve some of the existing results in the literature. To finish we consider several test problems.

Keywords Generalized gradient · Constraint qualifications · Lipschitzian problems

1 Introduction

In this paper, we consider the following Lipschitzian mathematical programming problem:

$$(P) \qquad \min f(x) + \phi(x), \qquad \text{s.t. } g(x) \leq 0, \quad x \in C, \tag{1}$$

where $f, \phi : \mathbb{R}^n \to \mathbb{R}$, $g = (g_1, g_2, \ldots, g_m) : \mathbb{R}^n \to \mathbb{R}^m$, g, f are assumed to be differentiable, ϕ is locally Lipschitzian continuous on C (see the definition in (1)) and C is a convex subset of $\mathbb{R}^n$.

In order to derive more important necessary optimality conditions, constraint qualifications are needed. There are six constraint qualifications in the book of Mangasarian

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(9), viz:

(CQ_1) the Kuhn–Tucker constraint qualification at $\bar{x}$;
(CQ_2) the Arrow–Hurwicz–Uzawa constraint qualification at $\bar{x}$;
(CQ_3) the reverse convex constraint qualification at $\bar{x}$;
(CQ_4) the Slater constraint qualification on C;
(CQ_5) the Karlin constraint qualification on C;
(CQ_6) the strict constraint qualification on C;

where C is an open subset of $\mathbb{R}^n$, $\bar{x} \in C$. Also, there is another constraint qualification called calmness in Clarke (1). Readers are referred to [9, Chapt. 7] for the relationships between (CQ_1)–(CQ_6). In view of the relationships, we need only to establish the results under the Kuhn–Tucker constraint qualification and the Arrow–Hurwicz–Uzawa constraint qualification. However, the Kuhn–Tucker constraint qualification or the Arrow–Hurwicz–Uzawa constraint qualification doesn't imply calmness constraint qualification, and calmness constraint qualification doesn't imply the Kuhn–Tucker constraint qualification or the Arrow–Hurwicz–Uzawa constraint qualification either.

Example 1.1 This example illustrates that the Kuhn–Tucker constraint qualification or the Arrow–Hurwicz–Uzawa constraint qualification doesn't imply calmness constraint qualification. Let

$$
\begin{aligned}
(Q_1) \quad & \min f(x) = -x, \\
& \text{s.t. } g_1(x) = x + x^3 \leq 0, \\
& \qquad g_2(x) = -x^2 \leq 0, \\
& \qquad x \in C = \{x \in \mathbb{R} : \ x \leq 1/2\}.
\end{aligned}
$$

Obviously, the feasible set is $S = \{x \in \mathbb{R} : \ x \leq 0\}$, $\bar{x} = 0$ is an optimal solution of (Q_1), and $\nabla g_1(\bar{x}) = 1$, $\nabla g_2(\bar{x}) = 0$.

1. Suppose that $d \in \mathbb{R}$, then the inequalities $d^T \nabla g_i(\bar{x}) \leq 0,\ i = 1, 2$ have the solution $\{d \in \mathbb{R} : \ d \leq 0\}$. Define a differentiable function defined on [0, 1]:

$$e(t) = \bar{x} + \lambda t d \quad \text{for some } \lambda > 0.$$

   We have $e(0) = \bar{x} = 0, e(t) \subseteq S$, and $\dot{e}(0) = \lambda d$. Then (Q_1) satisfy the Kuhn–Tucker constraint qualification at $\bar{x}$. Similarly, it is easy to verify that (Q_1) satisfy the Arrow–Hurwicz–Uzawa constraint qualification at $\bar{x}$. We leave the details for the readers.
2. For any integer $k > 0$, denote $p_k = (0, \frac{1}{k}), x^k = \sqrt{\frac{1}{k}}$. Then $p_k \to (0,\ 0)^T, x^k \to 0$ as $k \to +\infty$, and $x^k \in (\bar{x} + \sqrt{\frac{1}{k}} B) \cap \{x \in C : \ x + x^3 \leq 0,\ -x^2 + \frac{1}{k} \leq 0\}$(where B is a unit circle in $\mathbb{R}$). However,

$$\frac{f(x^k) - f(\bar{x})}{|p_k|} = \frac{-\sqrt{1/k}}{\frac{1}{k}} = -\sqrt{k} \to -\infty \quad (k \to \infty).$$

   So (Q_1) doesn't satisfy the calmness constraint qualification at $\bar{x}$.

Example 1.2 This example illustrates that calmness constraint qualification doesn't imply the Kuhn–Tucker constraint qualification or the Arrow–Hurwicz–Uzawa constraint qualification. Let

$$(Q_2) \quad \min f(x_1,\ x_2) = -x_1, \\ \text{s.t. } g(x_1,\ x_2) = x_1^{1/3} \le 0, \\ x \in \mathbb{R}^2.$$

Obviously, $\bar{x} = (0,\ 0)^T$ is an optimal solution of (Q_2), and $f(\bar{x}) = 0$. Since $g(x_1, x_2)$ is not differentiable at $\bar{x}$, (Q_2) doesn't satisfy the Kuhn–Tucker constraint qualification at $\bar{x}$ or the Arrow–Hurwicz–Uzawa constraint qualification at $\bar{x}$. However, $V(0) = 0$, $V(p) = -p^3$,

$$\liminf_{p\to 0} \frac{V(p) - V(0)}{|p|} = \liminf_{p\to 0} \frac{-p^3}{|p|} = 0 > -\infty.$$

So (Q_2) satisfy calmness constraint qualification at $\bar{x}$.

The necessary optimality conditions for problem (P) can be obtained under calmness constraint qualification [1, Chap. 6]. So in the present paper, it is of interest to find more practical constraint qualifications under which a local minimizer $\bar{x}$ of problem (P) is a (generalized) Kuhn–Tucker point of the problem; i.e., there is $\bar{\lambda} \in \mathbb{R}^m_+$ such that

$$0 \in \nabla f(\bar{x}) + \partial\phi(\bar{x}) + \nabla g(\bar{x})\bar{\lambda} + N_C(\bar{x}),$$

$$g(\bar{x})\bar{\lambda} = 0,$$

where ∂,$N_C(\bar{x})$ denote the generalized gradient operator in the sense of Clarke (1) and the normal cone to C at $\bar{x}$, respectively, and

$$\nabla g(\bar{x}) = (\nabla g_1(\bar{x}), \ldots, \nabla g_m(\bar{x})) \in \mathbb{R}^{n\times m}.$$

A class of nondifferentiable problems, which has been studied extensively, is as follows:

$$(P_1) \qquad \min f(x) + \phi(x), \qquad \text{s.t. } g(x) \le 0, \quad x \in C. \tag{2}$$

Where C and g are as above and $f,\ \phi : \mathbb{R}^n \to \mathbb{R}$, f is assumed to be differentiable, and ϕ is a proper convex function on $\mathbb{R}^n$.

When $\phi(x) = (x^T Bx)^{1/2}$ and $C = \mathbb{R}^n$, Mond (2) proposed the problem and got a necessary optimality condition under a certain complicated constraint qualification. Later, the problem was generalized by Aggarwal and Saxena (3) to fractional programming, and then by Singh (4) and Lai et al. (5) to minimax fractional programming. Corresponding necessary conditions were obtained under the constraint qualifications of the same type as given in (2). Based on the necessary conditions, sufficient conditions and Wolfe-type duality were considered in the above-mentioned papers. Of course, if in problem proposed in Mond (2) both f and g are convex functions (not necessarily differentiable), then a necessary and sufficient optimality conditions can be obtained under the slater constraint qualification (see Schechter (6), for example). Recently, Xu (7) obtained the Kuhn–Tucker type necessary optimality conditions for problem (P_1) under the conditions of a generalized Kuhn–Tucker constraint qualification or a generalized Arrow–Hurwicz–Uzawa constraint qualification.

In order to improve the last situations we combine the arguments and then consider the programming problem (P). Moreover, notice that (1) generalizes the last situations without increasing the operational cost since if $\phi = 0$ is taken, the classical differentiable programming is obtained; if $f = 0$, the Lipschitzian programming is obtained; as in (8), a convex function is a locally Lipschitzian function on its effective domain, consequently, (1) can be seen as a generalization of (2). And the results obtained in (7) can also be generalized correspondingly.

The remainder of this paper is organized as follows. In Sect. 2, we present some useful lemmas by which our main results can be proved easily. In Sect. 3, we establish the main results: a necessary optimality condition for problem (P) (Theorem 3.1); and a necessary optimality condition for a special case of problem (P) in which the convex set C is given explicitly by a set of convex inequalities (Theorem 3.2). We propose in Sect. 4 some theoretical results with the application of the main results in this paper and show the case when the set C is open is a special one of our results.

2 Preliminaries

In order to derive the Kuhn–Tucker type necessary optimality conditions, we need the following important lemmas:

Lemma 2.1 *Let f be a locally Lipschitzian function on a convex set C and let $\bar{x} \in C$. If for any $x \in C$, there is a $\xi_x \in \partial f(\bar{x})$ such that $\langle \xi_x, x - \bar{x}\rangle \geq 0$, then there is a $\bar{\xi} \in \partial f(\bar{x})$ such that*

$$\langle \bar{\xi}, x - \bar{x}\rangle \geq 0 \quad \textit{for all } x \in C.$$

Proof By Proposition 2.1.2 in (1), we have

$$f^\circ(\bar{x}; v) = \max\{\langle \xi,\ v\rangle : \xi \in \partial f(\bar{x})\} \quad \text{for all } v \in \mathbb{R}^n.$$

By Theorem 4.2.5 in (8), we know that $f^\circ(\bar{x}; v)$ is a convex function of v. Hence by the hypothesis of the lemma

$$\min_{y\in C} f^\circ(\bar{x}; y - \bar{x}) = \min_{y\in C} \max\{\langle \xi,\ y - \bar{x}\rangle : \xi \in \partial f(\bar{x})\} = 0.$$

Without loss of generality, we can reduce the argument to the case where $\bar{x} = 0$, $f^\circ(0; 0) = 0$, and consequently

$$\min_{y\in C} f^\circ(\bar{x}; y - \bar{x}) = f^\circ(0; 0) = 0. \tag{3}$$

Let us consider now the convex sets

$$C_1 = \{(v, \mu) \in \mathbb{R}^{n+1} | \mu \geq f^\circ(0; v)\}, \tag{4}$$
$$C_2 = \{(v, \mu) \in \mathbb{R}^{n+1} | v \in C, \mu \leq 0\}. \tag{5}$$

According to Lemma 7.3 in (6), we have

$$\text{ri}C_1 = \{(v, \mu) \in \mathbb{R}^{n+1} | \mu > f^\circ(0; v)\},$$

$$\text{ri}C_2 = \{(v, \mu) \in \mathbb{R}^{n+1} | v \in \text{ri}C, \mu < 0\},$$

where ri$C$ denotes the relative interior of the convex set C(see the definition in (8)). Since the minimum of $f^\circ(0; \cdot)$ is 0, it follows that $(\text{ri}C_1) \cap (\text{ri}C_2) = \emptyset$. Hence C_1 and C_2 can be separated properly by some hyperplane in $\mathbb{R}^{n+1}$(Theorem 11.3 in (8)). The separating hyperplane cannot be vertical, for if it were its image under the projection $(x, \mu) \to x$ would be a hyperplane in $\mathbb{R}^n$ separating $\mathbb{R}^n$ and ri$C$ properly, and this is impossible because $(\mathrm{ri}C) \cap \mathbb{R}^n \neq \emptyset$. The separating hyperplane must therefore be the graph of an affine function on $\mathbb{R}^n$, in fact a linear function since C_1 and C_2 have the origin of $\mathbb{R}^{n+1}$ in common. Thus there is an $\bar{\xi} \in \mathbb{R}^n$ such that

$$\mu \geq \langle \bar{\xi}, v \rangle \quad \text{for all } (v, \mu) \in C_1, \tag{6}$$

$$\mu \leq \langle \bar{\xi}, v \rangle \quad \text{for all } (v, \mu) \in C_2. \tag{7}$$

By (3), we have

$$f^\circ(0; v) \leq f^\circ(0; v) - f^\circ(0; 0) \quad \text{for all } v \in \mathbb{R}^n$$

from (4), thus $(v, f^\circ(0; v) - f^\circ(0; 0)) \in C_1$ for all $v \in \mathbb{R}^n$. By (6), it follows that

$$f^\circ(0; v) - f^\circ(0; 0) \geq \langle \bar{\xi}, v - 0 \rangle \quad \text{for all } v \in \mathbb{R}^n. \tag{8}$$

Similarly, by (5), $(v, 0) \in C_2$, for all $v \in C$. From (7) we obtain

$$0 \leq \langle \bar{\xi}, v \rangle \quad \text{for all } v \in C. \tag{9}$$

Combining (8) and (9), it follows that

$$\bar{\xi} \in \partial f(0), \qquad \langle \bar{\xi}, v \rangle \geq 0 \quad \text{for all } v \in C$$

and the proof is complete. □

Remarks The property described in Lemma 2.1 can be used to linearize the non-smooth programming problem, just like the differentiability property of functions.

Lemma 2.2 *Let C be a convex set and let $\bar{x} \in C$, then $N_{\{\bar{x}\} \cup \mathrm{ri}C}(\bar{x}) = N_C(\bar{x})$.*

Proof First we have $N_{\{\bar{x}\} \cup \mathrm{ri}C}(\bar{x}) \supseteq N_C(\bar{x})$ by definition. If $y \in N_{\{\bar{x}\} \cup \mathrm{ri}C}(\bar{x})$, we have

$$y^T(x - \bar{x}) \leq 0 \quad \text{for all } x \in \{\bar{x}\} \cup \mathrm{ri}C.$$

If $\bar{x} \in \mathrm{ri}C$,

$$\lambda\bar{x} + (1 - \lambda)z \in \mathrm{ri}C \quad \text{for all } z \in C \quad \text{for all } \lambda \in (0\,,\,1),$$

$$y^T[\lambda\bar{x} + (1 - \lambda)z] = (1 - \lambda)y^T(z - \bar{x}) \leq 0.$$

If $\bar{x} \in C \setminus (\mathrm{ri}C) \subseteq (\mathrm{cl}C) \setminus (\mathrm{ri}C)$, then we have

$$\lambda x + (1 - \lambda)\bar{x} \in \mathrm{ri}C \quad \text{for all } x \in (\mathrm{ri}C).$$

Hence

$$y^T[\lambda x + (1 - \lambda)\bar{x} - \bar{x}] = \lambda y^T(x - \bar{x}) \leq 0.$$

Consequently $y \in N_C(\bar{x}), N_{\{\bar{x}\} \cup \mathrm{ri}C}(\bar{x}) \subseteq N_C(\bar{x})$. So $N_{\{\bar{x}\} \cup \mathrm{ri}C}(\bar{x}) = N_C(\bar{x})$. The proof is completed. □

Lemma 2.3 (7) *Let C be a nonempty convex set of $\mathbb{R}^n$. Then, for any $x^\circ \in (\mathrm{cl}C) \setminus (\mathrm{ri}C)$, one has*

$$(\mathrm{ri}C - x^\circ) \cap T_{(\mathrm{aff}C)\setminus(\mathrm{ri}C)}(x^\circ) = \varnothing,$$

where $T_{(\mathrm{aff}C)\setminus(\mathrm{ri}C)}(x^\circ)$ denotes the tangent cone to $(\mathrm{aff}C)\setminus(\mathrm{ri}C)$ at x° in the sense of Clarke (1).

Lemma 2.4 (8) *Let f be a proper convex function on $\mathbb{R}^n$, and let S be an any bounded closed subset of* ri(dom$f$)*, then f is Lipschitzian on S.*

3 Main results

For reader's convenience, we write the problem defined in Sect. 1

$$(P) \qquad \min f(x), \qquad \text{s.t. } g(x) \le 0, \quad x \in C.$$

Denote

$$X = \{x \in \mathbb{R}^n | g(x) \le 0\},$$

$$I = I(\bar{x}) = \{i | g_i(x) = 0, i = 1, 2, \ldots, m\}.$$

Let g_I be the row vector whose components are g_i, $i \in I$ and let ∇g_I be the matrix whose ith column is ∇g_i for $i \in I$. Denote

$$Z(\bar{x}) = \{x \in \mathbb{R}^n | \nabla g_I(\bar{x})(x - \bar{x}) \le 0\}$$

with the convention that if $I = \varnothing$ then $Z(\bar{x}) = \mathbb{R}^n$.

Definition 3.1 (8) g is said to satisfy the generalized Kuhn–Tucker constraint qualification at $\bar{x} \in X \cap C$ if for each $x \in (\mathrm{ri}C) \cap Z(\bar{x})$, there exists a differentiable function α defined on $[0\,,\,1]$ with range in $\mathbb{R}^n$ such that

$$\alpha(0) = \bar{x}, \qquad \alpha(t) \in X \cap (\mathrm{aff}C), \quad t \in [0\,,\,1] \tag{10}$$

and for some $\delta > 0$

$$\frac{\mathrm{d}\alpha(0)}{\mathrm{d}t} = \delta(x - \bar{x}). \tag{11}$$

Note that, in Definition 3.1, we use the condition $\alpha(t) \in X \cap (\mathrm{aff}C)$ for $t \in [0\,,\,1]$ instead of the condition $\alpha(t) \in X \cap C$ for $t \in [0\,,\,1]$ given in (9). The former is weaker than the latter in general; if C has a nonempty interior (not necessarily convex), then $\mathrm{aff}C = \mathbb{R}^n$ and the former becomes $\alpha(t) \in X$ for $t \in [0\,,\,1]$.

Definition 3.2 (8) g is said to satisfy the generalized Arrow–Hurwicz–Uzawa constraint qualification at $\bar{x} \in X \cap C$ if

$$\nabla g_W(\bar{x})^T(x - \bar{x}) < 0, \quad \nabla g_V(\bar{x})^T(x - \bar{x}) \le 0 \tag{12}$$

has a solution

$$x \in (\mathrm{aff}C), \tag{13}$$

where

$$V = \{i : g_i(\bar{x}) = 0,\ g_i \text{ is concave}\},$$

$$W = \{i : g_i(\bar{x}) = 0,\ g_i \text{ is not concave}\}.$$

We also note that, in Definition 3.2, if C has a nonempty interior, then the condition that (12) has a solution $x \in \text{aff}C$ becomes the condition that (12) has a solution $x \in \mathbb{R}^n$, which coincides with the corresponding one in (9).

Theorem 3.1 *Let $\bar{x}$ solve problem (P), and let g satisfy*

1. *the generalized Kuhn–Tucker constraint qualification at $\bar{x}$ in Definition* 3.1*, or*
2. *the generalized Arrow–Hurwicz–Uzawa constraint qualification at $\bar{x}$ in Definition* 3.2*.*
   If $(\text{ri}C) \cap Z(\bar{x}) \neq \varnothing$*, then there exist* $\bar{\lambda} \in \mathbb{R}^m_+$ *such that*

$$0 \in \nabla f(\bar{x}) + \partial\phi(\bar{x}) + \nabla g(\bar{x})\bar{\lambda} + N_c(\bar{x}), \tag{14}$$

$$g(\bar{x})\bar{\lambda} = 0. \tag{15}$$

Proof

1. Take $x^* \in (\text{ri}C) \cap Z(\bar{x})$. First, as in Definition 3.1 we have

$$\alpha(t) \in X \cap (\text{aff}C), \quad t \in [0\,,\,1]. \tag{16}$$

Next, we show that there exists a $\varepsilon : 0 < \varepsilon < 1$ such that

$$\alpha(t) \in \text{ri}C, \quad t \in (0\,,\,\varepsilon]. \tag{17}$$

If $\bar{x} \in (\text{ri}C)$, there exists a $\bar{\varepsilon} > 0$ such that

$$(\bar{x} + \bar{\varepsilon}K) \cap (\text{aff}C) \subseteq \text{ri}C. \tag{18}$$

Combining (16) and (18) we can conclude that (17) holds true. So, we suppose that $\bar{x} \in C\backslash(\text{ri}C) \subseteq (\text{cl}C)\backslash(\text{ri}C)$. Suppose on the contrary that there exist $\{t_k\}$ with $t_k > 0, t_k \to 0$ such that $\alpha(t_k) \in (\text{aff}C)\backslash(\text{ri}C)$. Then, it follows from (11) that

$$\frac{1}{t_k}(\alpha(t_k) - \alpha(0)) \to \delta(x^* - \bar{x})$$

or

$$\frac{1}{\delta t_k}(\alpha(t_k) - \alpha(0)) \to \delta(x^* - \bar{x}),$$

implying $x^* - \bar{x} \in T_{(\text{aff}C)\backslash(\text{ri}C)}(\bar{x})$. By Lemma 2.1 we have $x^* \notin \text{ri}C$, which contradicts with $x^* \in \text{ri}C$. The combination of (16) and (17) gives that there exists a $\varepsilon : 0 < \varepsilon < 1$ such that

$$\alpha(t) \in X \cap C, \quad t \in [0\,,\,\varepsilon]. \tag{19}$$

Since $\bar{x}$ solves problem (P) and $\alpha(0) = \bar{x}$, it follows from (19) that $t = 0$ is a solution of the following problem:

$$\min_{t\in[0\,,\,\varepsilon]} \omega(t) = f(\alpha(t)) + \phi(\alpha(t)). \tag{20}$$

Then, by the corollary of Proposition 2.4.3 in (1), we have $\eta \geq 0$ for some $\eta \in \partial\omega(0)$, where $\partial\omega(0)$ is the generalized gradient of the function $\omega(t)$ at $t = 0$ in the sense of Clarke (1). Using Proposition 2.3.3, Theorem 2.3.10 in (1) and (11) to compute the generalized gradient of $\omega(t)$ at $t = 0$, we have that there exist $\xi = \xi(\bar{x}, x^*) \in \partial\phi(\bar{x})$ such that

$$\nabla f(\bar{x})^T(x^* - \bar{x}) + \xi^T(x^* - \bar{x}) \geq 0. \tag{21}$$

Obviously both ri$C$ and $Z(\bar{x})$ are both convex sets, then (ri$C$) $\cap Z(\bar{x})$is a convex set. By Lemma 2.1 there exist $\bar{\xi} \in \partial\phi(\bar{x})$ such that

$$\nabla f(\bar{x})^T(x^* - \bar{x}) + \bar{\xi}^T(x^* - \bar{x}) \geq 0 \quad \text{for all } x^* \in (\text{ri}C) \cap Z(\bar{x}). \tag{22}$$

Since $x^* \in (\text{ri}C) \cap Z(\bar{x})$ is arbitrarily chosen,we also have

$$\nabla f(\bar{x})^T(x - \bar{x}) + \bar{\xi}^T(x - \bar{x}) \geq 0 \quad \text{for all } x \in (\text{ri}C) \cap Z(\bar{x}). \tag{23}$$

Hence, the system

$$\nabla f(\bar{x})^T(x - \bar{x}) + \bar{\xi}^T(x - \bar{x}) < 0, \qquad \nabla g_I(\bar{x})^T(x - \bar{x}) \leq 0 \tag{24}$$

has no solution $x \in \text{ri}C$.
By (ri$C$) $\cap Z(\bar{x}) \neq \varnothing$ and alternative theorem, there exists a vector $\bar{\lambda}_I \geq 0$ such that

$$\nabla f(\bar{x})^T(x - \bar{x}) + \bar{\xi}^T(x - \bar{x}) + (\bar{\lambda}_I)^T\nabla g_I(\bar{x})^T(x - \bar{x}) \geq 0, \quad x \in \text{ri}C.$$

Denote $F(x) = \nabla f(\bar{x})^T(x - \bar{x}) + \bar{\xi}^T(x - \bar{x}) + (\bar{\lambda}_I)^T\nabla g_I(\bar{x})^T(x - \bar{x})$. Then $F(x)$ has a global minimum point $\bar{x}$ on the convex set $\{\bar{x}\} \cup \text{ri}C$, then by the corollary of Proposition 2.4.3 in (1), we have

$$0 \in \nabla f(\bar{x})\bar{\xi} + \nabla g_I(\bar{x})\bar{\lambda}_I + N_{\{\bar{x}\}\cup \text{ri}C}(\bar{x}).$$

By Lemma 2.2 and $\bar{\xi} \in \partial\phi(\bar{x})$,

$$0 \in \nabla f(\bar{x}) + \partial\phi(\bar{x}) + \nabla g_I(\bar{x})\bar{\lambda}_I + N_C(\bar{x}).$$

If $i \notin I(\bar{x})$, take $\bar{\lambda}_i = 0$, then there exist $\bar{\lambda} \in \mathbb{R}^m_+$ such that (14) and (15) hold true.

2. Take $x^* \in (\text{ri}C) \cap Z(\bar{x})$. Choose some $\hat{x} \in \text{aff}C$ satisfying (12). Define

$$\alpha_s(t) = \bar{x} + t[(x^* - \bar{x}) + s(\hat{x} - \bar{x})], \tag{25}$$

where s and t are scalars. Clearly, $\alpha_s(0) = \bar{x}$.

$$\alpha_s(t) \subseteq \text{aff}C \quad \text{for all } t \quad \text{for all } s. \tag{26}$$

So $x^*, \bar{x}, \hat{x}$ belong to aff$C$, and

$$\frac{\mathrm{d}\alpha_s(0)}{\mathrm{d}t} = (x^* - \bar{x}) + s(\hat{x} - \bar{x}). \tag{27}$$

We are going to show that for any $s > 0$ there exists a $\varepsilon_1 = \varepsilon_1(s) > 0$ such that

$$\alpha_s(t) \in X, \quad t \in [0\ ,\ \varepsilon_1]. \tag{28}$$

For each $i \in V$

$$g_i(\alpha_s(t)) = g_i(\alpha_s(t)) - g_i(\alpha_s(0)) \leq \nabla g_i(\bar{x})^T[t(x^* - \bar{x}) + ts(\hat{x} - \bar{x})] \leq 0.$$

For each $i \in W$, since

$$\frac{\mathrm{d}g_i(\alpha_s(0))}{\mathrm{d}t} = \nabla g_i(\bar{x})^T[(x^* - \bar{x}) + s(\hat{x} - \bar{x})] < 0.$$

We have

$$g_i(\alpha_s(t)) < 0 \quad \text{for small } t > 0. \tag{29}$$

Finally, for each $i \in \{1, \ldots, m\} \setminus (W \cup V)$, it is clear that (29) holds true since $g_i(\bar{x}) < 0$ and $\alpha_s(t)$ is continuous with respect to t. Hence, (28) is true. The combination of (26) and (28) gives

$$\alpha_s(t) \in X \cap (\mathrm{aff}C), \quad t \in [0\,,\ \varepsilon_1]. \tag{30}$$

Similarly to the arguments in the first two paragraphs in the proof of (1), we have that there exists a ε_2 and ε_3 such that

$$\alpha_s(t) \in \mathrm{ri}C, \quad t \in (0\,,\ \varepsilon_2], \quad s \in (0\,,\ \varepsilon_3]. \tag{31}$$

Set $\varepsilon_4 = \min(\varepsilon_1, \varepsilon_2)$. We have that, for any fixed $s \in (0\,,\ \varepsilon_3]$,

$$\alpha_s(t) \in X \cap C, \quad t \in [0\,,\ \varepsilon_4]. \tag{32}$$

Similarly to (21), there exist $\xi = \xi(x^*, \bar{x}, \hat{x}, s) \in \partial\phi(\bar{x})$ such that

$$\nabla f(\bar{x})^T[(x^* - \bar{x}) + s(\hat{x} - \bar{x})] + \xi^T[(x^* - \bar{x}) + s(\hat{x} - \bar{x})] \geq 0, \quad s \in (0\,,\ \varepsilon_3] \tag{33}$$

and then, by letting $s \to 0$ in (33),

$$\nabla f(\bar{x})^T(x^* - \bar{x})\xi^T(x^* - \bar{x}) \geq 0.$$

This is exactly inequality(21). The rest of the proof is similar, and the proof is complete. □

If, in problem (P), we have,

$$C = \{x \in \mathbb{R}^n :\ h(x) \leq 0\},$$

where h is a vector convex function from $\mathbb{R}^n$ to $\mathbb{R}^p$, then it is easy to verify that

$$\mathrm{ri}C = \mathrm{int}C = \{x \in \mathbb{R}^n :\ h(x) < 0\},$$

provided that the inequality $h(x) < 0$ has a solution. If we keep in mind the remarks right below Definitions 3.1 and 3.2, the following result is at hand, which may be viewed as a necessary condition by using a mixed-type constraint qualification: the Kuhn–Tucker constraint qualification plus the Slater constraint qualification, or the Arrow–Hurwica–Uzawa constraint qualification plus the Slater constraint qualification.

Theorem 3.2 *Consider the following problem:*

$$(P') \qquad \min f(x) + \phi(x), \qquad s.t.\ g(x) \leq 0, \quad h(x) \leq 0,$$

where f, ϕ, and g are as given in problem (P), h is a convex vector function from $\mathbb{R}^n$ to $\mathbb{R}^p$. Let $\bar{x}$ solve problem (P'), and let g satisfy any of the following two constraint qualifications:

1. *the Kuhn–Tucker constraint qualification in Definition* 3.1 *with condition* (10) *being replaced by the condition* $\alpha(t) \in X$ *for* $t \in [0\ ,\ 1]$*;*
2. *the Arrow–Hurwica–Uzawa constraint qualification in Definition* 3.2 *with condition* (13) *being replaced by the condition* $x \in \mathbb{R}^n$*.*

If

$$\{x \in \mathbb{R}^n :\ h(x) < 0\} \cap Z(\bar{x}) \neq \emptyset, \tag{34}$$

then there exist $\bar{\lambda} \in \mathbb{R}^m_+$, $\bar{\mu} \in \mathbb{R}^p_+$ *such that*

$$0 \in \nabla f(\bar{x}) + \partial\phi(\bar{x}) + \nabla g(\bar{x})\bar{\lambda} + \partial h(\bar{x})\bar{\mu},$$

$$g(\bar{x})\bar{\lambda} = 0, \quad h(\bar{x})\bar{\mu} = 0,$$

where

$$\partial h(\bar{x}) = (\partial h_1(\bar{x}), \ldots, \partial h_p(\bar{x})).$$

Proof Apply Theorem 21.2 in (8) to (23) with the expression $x \in (\mathrm{ri}C) \cap Z(\bar{x})$ being replaced by the corresponding inequalities, and use condition (34). The proof is complete. □

4 Special cases

Suppose now that the set C in problem (P) is open (not necessarily convex). Then $\mathrm{ri}C = \mathrm{int}C$, equation $(\mathrm{ri}C) \cap Z(\bar{x}) \neq \varnothing$ is automatically satisfied and $N_C(\bar{x}) = 0$. Choose an open ball C_1 included in C with center $\bar{x}$. Replacing C by C_1 in problem (P) to get a new problem, denoted by (P_2). We see that, via writing a theorem for problem (P_2) similarly to Theorem 3.1, the constraint qualification reduce to the corresponding classical ones.

For problem (P_1), replacing C by $(\mathrm{ri}C) \cup \bar{x}$ to get a new problem denoted (P_3), that $\bar{x}$ solves problem (P_1) implies it solves problem (P_3). By Lemma 2.4 $f(x) + \phi(x)$ is a locally Lipschitzian function on $(\mathrm{ri}C) \cup \bar{x}$. If g satisfy the hypothesis of Theorem 3.1, then there exist $\bar{\lambda} \in \mathbb{R}^m_+$ such that

$$0 \in \nabla f(\bar{x}) + \partial\phi(x) + \nabla g(\bar{x})\bar{\lambda} + N_{(\mathrm{ri}C)\cup\bar{x}}(\bar{x}),$$

$$g(\bar{x})\bar{\lambda} = 0.$$

By Lemma 2.2 we have

$$0 \in \nabla f(\bar{x}) + \partial\phi(x) + \nabla g(\bar{x})\bar{\lambda} + N_C(\bar{x}),$$

$$g(\bar{x})\bar{\lambda} = 0.$$

Hence problem (P_1) is a special case of problem (P).

Suppose now $C = \mathbb{R}^n$ in problem (P), we get a new programming denoted (P_4). It is easy to check that the problem proposed in (2) is a special case of problem (P_4). The constraint qualifications in this paper obviously are the extension of the classical

Kuhn–Tucker constraint qualification and Arrow–Hurwicz–Uzawa constraint qualification to nondifferentiable programming, which is more practical and simpler than the constraint qualification in (2,4).

Now we point out that Theorem 3.1 applies in order to derive the necessary conditions for the fractional programming as follows:

$$(P_5) \qquad \min \frac{f_1(x)+\phi_1(x)}{f_2(x)+\phi_2(x)}, \qquad s.t.\ g(x) \le 0, \quad x \in C,$$

where g, C are as (P), $f_1(x), f_2(x)$ are assumed differentiable, $\phi_1(x), \phi_2(x)$ are Lipschitzian functions on $\mathbb{R}^n$, and for all $x \in C$, $f_2(x) + \phi_2(x) > 0$. It is easy to see that $\frac{f_1(x)+\phi_1(x)}{f_2(x)+\phi_2(x)}$ is locally Lipschitzian on C. If $\bar{x}$ solves (P_5) and g satisfy the hypothesis of Theorem 3.1, denote $\vartheta(\bar{x}) = (f_1(\bar{x}) + \phi_1(\bar{x}))/(f_2(\bar{x}) + \phi_2(\bar{x}))$, then there exist $\bar{\lambda} \in \mathbb{R}^m_+$ such that

$$0 \in \nabla f_1(\bar{x}) + \partial\phi_1(x) - \vartheta(\bar{x})[\nabla f_2(\bar{x}) + \partial\phi_2(x)] + \nabla g(\bar{x})\bar{\lambda} + N_C(\bar{x}),$$

$$g(\bar{x})\bar{\lambda} = 0,$$

which include corresponding necessary conditions in (4).

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